Admissible consensus for heterogeneous descriptor multi-agent systems

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This paper focuses on the admissible consensus problem for heterogeneous descriptor multi-agent systems. Based on algebra, graph and descriptor system theory, the necessary and sufficient conditions are proposed for heterogeneous descriptor multi-agent systems achieving admissible consensus. The provided conditions depend on not only the structure properties of each agent dynamics but also the topologies within descriptor multi-agent systems. Moreover, an algorithm is given to design the novel consensus protocol. A numerical example demonstrates the effectiveness of the proposed design approach.

Keywords: Admissible consensus, heterogeneous descriptor multi-agent systems, consensus protocol

1. Introduction

In recent years, distributed coordination control for multi-agent systems has attracted considerable attention due to board applications including flocking problem, formation problem, distributed sensor networks and congestion control in communication networks, and so on (Feng & Hu, 2014; Guo et al., 2014; Yu et al., 2013; Zhu et al., 2013). Consensus problem is one of the most fundamental distributed coordination control problems (Pan et al., 2014; Zhang & Tian, 2014; Zheng, 2014). Consensus means that multiple agents reach an agreement on a common value which might be, for example, heading direction in flocking behavior, average in distributed computation, or the altitude in multi-spacecraft alignment (Lin & Jia, 2010; Nguyen & Tran, 2013; Sedziwy, 2014). In addition, it is worth mentioning that descriptor systems (also referred to as singular systems, semi-state systems, generalized state-space systems, implicit systems or differential-algebraic systems) have provided a more natural description of dynamical systems than state space systems (Boughari & Radhy, 2007; Cong, 2014; Huang, 2014; Su et al., 2013). Descriptor systems, as a form of differential algebraic equations, are better able to maintain the physical characteristics of systems, especially in some coupled systems where constraints characterized by algebraic equations indeed exist between some physical quantities (Hsieh, 2014; Li et al., 2014; Zhang et al., 2014). Meanwhile, many practical systems can not be described by normal systems but descriptor systems. Descriptor systems have a more comprehensive background, such as power systems, circuit systems, aerospace engineering, chemical processes, social economic systems, network analysis, biological systems, and so on.

In summary, the study based on combining multi-agent systems with descriptor systems has important theory significance and practical application value. Hence this paper puts an intensive study on admissible consensus problems with multi-agent systems composed of descriptor systems as the research object. The concept of descriptor multi-agent systems has been introduced in Yang & Liu (2012). Taking a robot system for example (Duan, 2010), consider a three-link manipulator shown in Fig. 1 whose task is to clean the region between points A and B. The robot fulfills the

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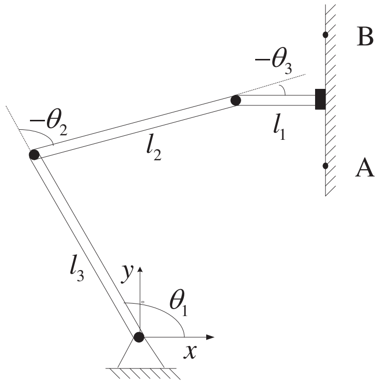

Figure 1. A three-link planar manipulator (Duan, 2010)

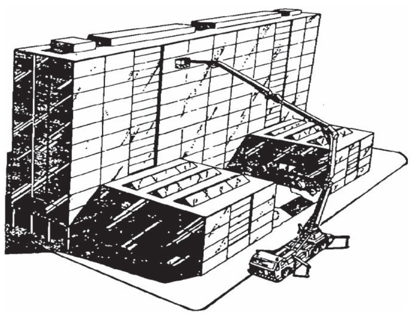

Figure 2. A cleaning mobile manipulator (Duan, 2010)

task with the given time by moving the end-effector of the manipulator from point A to point B repeatedly with the specified contact force. In fact, this is a simplified model and task description of the large mobile manipulator cleaning the facade of a large building illustrated in Fig. 2. Assume that the cleaning flat surface is a rigid body and the third arm is a smooth and rigid plate. By adding the constraint equations and the corresponding generalized constrained force to the motion equation of a free robot, the motion of the constrained robots can be easily modeled as a descriptor system. The specific details of deduction have been shown in Duan (2010). Thus, the descriptor multi-agent system is made of several or many descriptor systems like three-link manipulators over networks. In this example, achieving consensus means that state differences between different manipulators tend to zero, respectively. Roughly speaking, these manipulators move in the very similar trajectories, but these trajectories do not overlap.

In the last two decades, many results of state space systems have been extended to descriptor systems (Dong, 2014; Li & Zhang, 2012; Wang et al., 2014; Yang et al., 2010). However, rare works have been published to deal with admissible consensus of descriptor multi-agent systems. For homogenous descriptor multi-agent systems with fixed topologies, Yang & Liu (2012) have provided the necessary and sufficient consensus conditions with respect to a set of admissible consensus protocols. Xi et al. (2012) have dealt with consensus problem for linear time-invariant homogenous descriptor swarm system. For singular high-order homogenous multi-agent systems with switching topologies, guaranteed-cost consensus problems have been investigated in Xi et al. (2014). And linear matrix inequality conditions have been provided for consensualization and guaranteed-cost consensus, respectively. Based on the networked predictive control scheme, static output feedback and observer, Yang & Liu (2014) has proposed protocols to guarantee that the studied descriptor system achieve consensus, and the protocol also can eliminate the negative effect of networked delays. It is regrettable that Xi et al. (2012, 2014); Yang & Liu (2012) and Yang & Liu (2014) do not consider heterogeneous descriptor multi-agent systems. Due to the impact of external environment, specific task distribution or other factors, the characteristics and dynamic models of coupled agents may be different, and these systems are called heterogeneous multi-agent systems. Comparing the discussion of consensus problems for homogeneous multi-agent systems with that for heterogeneous ones, it is easy to see that the later one has more significant due to heterogeneous multi-agent systems have more extensive description in the real world. In this paper, the admissible consensus problem is studied for heterogeneous descriptor multi-agent systems with fixed topologies and agents described different dynamics. A novel protocol is proposed to solve the admissible consensus problem. Furthermore, the consensus algorithm is supplied to design the consensus protocol. The provided numerical example demonstrates the effectiveness of the proposed design approach.

The rest of this paper is organized as follows. Some preliminaries on descriptor system theory and the problem formulation are described in Section 2. In Section 3, admissible consensus analysis for heterogeneous multi-agent systems is discussed. Meanwhile, the proposed consensus protocol is designed. A numerical example will be presented in Section 4 to verify the feasibility and effectiveness of the theoretical results obtained in this paper. Section 5 is devoted to conclusions.

2. Preliminaries and problem formulation

2.1 *Preliminaries*

Throughout the paper, let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}^-$ and I_n denote the real plane, the complex plane, the open left-half complex plane and the identity matrix of order n, respectively. $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$ represent a set of all real matrices of dimension $m\times n$ and a set of all complex matrices of dimension $m\times n$, respectively. For the given vector x, $\|x\|$ stands for the Euclidean norm.

Definition 1: (Ben-Israel & Greville, 1974) Let $A\in\mathbb{C}^{m\times n}$. Then the matrix $X\in\mathbb{C}^{n\times m}$ satisfying the following four equations (usually called the Penrose conditions)

$$AXA=A,\ XAX=X,\ (AX)^*=AX,\ (XA)^*=XA$$

is called the Moore-Penrose inverse of A, and is denoted by $X=A^{\dagger}$.

Lemma 1: *(Ben-Israel & Greville, 1974) Let $A\in\mathbb{C}^{m\times n}$. The generalized inverse X satisfying the Penrose conditions is existent and unique. Moreover, if* $\mathrm{rank}A=n$, *then $A^{\dagger}A=I_n$; if* $\mathrm{rank}A=m$, *then $AA^{\dagger}=I_m$.*

Lemma 2: *(Horn & Johnson, 1990) For any matrix $A=[a_{ij}]$, B, C and D with appropriate dimensions, the Kronecker product of matrices A and B is defined as:*

$$A\otimes B=\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix},$$

it has the following properties:

$$(A\otimes B)(C\otimes D)=(AC)\otimes(BD),$$

$$A\otimes B+A\otimes C=A\otimes(B+C).$$

Definition 2: Let $E,A\in\mathbb{R}^{n\times n}$.

(i) (Wu & Zhou, 2007) The pair (E,A) is said to be regular if $\det(sE-A)$ is not identically zero for some $s\in\mathbb{C}$, where $\det(\cdot)$ represents determinant of a matrix;

(ii) (Xi et al., 2012) The pair (E, A) is said to be impulse free if (E, A) is regular and deg det$(sE - A) = \text{rank}E$ for $\forall s \in \mathbb{C}$, where deg$(\cdot)$ represents degree of a polynomial, respectively;

(iii) (Duan, 2010) The pair (E, A) is said to be stable if $\sigma(E, A) \subseteq \mathbb{C}^-$, where $\sigma(E, A) = \{\lambda | \lambda \in \mathbb{C},\ \lambda \text{ is finite}, \det(\lambda E - A) = 0\}$;

(iv) (Duan, 2010) The pair (E, A) is said to be admissible if (E, A) is impulse free and stable.

Remark 1: The regularity of descriptor systems can guarantee the existence and uniqueness of solutions (Duan, 2010).

Definition 3: (Duan, 2010) The regular descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t),$$

or (E, A, B) is called stabilizable if there exists a state feedback $u = Kx + v$ such that the resulted closed-loop system

$$E\dot{x} = (A + BK)x + Bv,$$

is stable, where $K \in \mathbb{R}^{r \times n},\ v \in \mathbb{R}^r$ is auxiliary input signal.

Lemma 3: *(Yang et al., 2004) Descriptor linear system*

$$E\dot{x}(t) = Ax(t)$$

is admissible (stable) if and only if the pair (E, A) is admissible (stable).

Lemma 4: *(Yang et al., 2004) For $E, A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times r}$ and $C \in \mathbb{R}^{m \times n}$, assume that (E, A) is impulse free, (E, A, B) is stabilizable, and (E, A, C) is detectable. Then Riccati equation*

$$A^T X + X^T A - X^T B B^T X + C^T C = 0, \tag{1a}$$

$$E^T X = X^T E \geq 0 \tag{1b}$$

has at least one admissible solution X, i.e., $(E, A - BB^T X)$ is admissible, where X is the maximum solution. Furthermore, the admissible solution X is unique in the sense of $E^T X$.

Lemma 5: *(Yang & Liu, 2012) If P is an admissible solution of Riccati equation (1) and (E, A, C) is detectable, then $\sigma\left(E,\ [A - (a + b\mathbf{i})BB^T P]\right) \subseteq \mathbb{C}^-(\mathbf{i}^2 = -1)$ for $\forall a \geq \frac{1}{2},\ b \in \mathbb{R}$.*

2.2 *Problem formulation*

Consider a class of heterogeneous descriptor multi-agent systems consisting of N agents indexed by $1, 2, \cdots, N$, respectively. The dynamics of the i-th agent is described as:

$$E\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t),\ i = 1,\ 2, \cdots\ N, \tag{2}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ are the state and control input of the i-th agent, respectively; $E, A_i \in \mathbb{R}^{n \times n},\ B_i \in \mathbb{R}^{n \times m}$; $\text{rank}E = r \leq n$. Assume that B_i has a full-row rank, $i = 1,\ 2, \cdots\ N$.

Remark 2: In the above descriptor multi-agent system, if matrix E is nonsingular, this descriptor multi-agent system becomes a normal multi-agent system. Therefore, from taking account of the wide rang of elements of matrix E, it is obtained that descriptor multi-agent systems are generalizations of normal multi-agent systems.

Regarding the above N agents as vertices, the topology relationship among them can be conveniently described by a directed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with a set of nodes $\mathcal{V} = \{1, 2, \cdots, N\}$ denoting the agents, a set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and a nonnegative weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$. In directed graph $\mathcal{G}$, a directed edge e_{ij} is denoted by the ordered pair of nodes (j, i), where node j and node i are called parent and child node, respectively. The neighbor set of the i-th agent is denoted by $N_i = \{j \in \mathcal{V} | e_{ij} \in \mathcal{E}\}$. The adjacency elements $a_{ii} = 0$, $a_{ij} > 0 \Leftrightarrow j \in N_i$ associated with e_{ij}, otherwise, $a_{ij} = 0$. Moreover, if there exists a node, called root, such that there is a directed path from this node to any other nodes, the graph is said to contain a directed spanning tree. The Laplacian matrix $\mathcal{L}_{\mathcal{G}} = [l_{ij}] \in \mathbb{R}^{N \times N}$ of the digraph $\mathcal{G}$ is defined as:

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum\limits_{k=1, k \neq i}^{N} a_{ik}, & i = j. \end{cases} \tag{3}$$

Clearly, all row-sums of $\mathcal{L}_{\mathcal{G}}$ are zero, which implies that $\mathcal{L}_{\mathcal{G}}$ has at least one zero eigenvalue and corresponding the right eigenvector $\mathbf{1}_N$, where $\mathbf{1}_N = [1 \ 1 \ \cdots 1]^T \in \mathbb{R}^N$. Meanwhile, Laplacian matrix $\mathcal{L}_{\mathcal{G}}$ has the following property.

Lemma 6: *(Ren & Beard, 2005) The Laplacian matrix $\mathcal{L}_{\mathcal{G}}$ of a directed graph $\mathcal{G}$ has at least one zero eigenvalue and all non-zero eigenvalues are in the open right-half plane. Furthermore, $\mathcal{L}_{\mathcal{G}}$ has exactly one zero eigenvalue if and only if $\mathcal{G}$ contains a directed spanning tree.*

Adopting state feedback is not only simple and feasible in terms of design, but can also improve the performance of systems effectively. Hence the state feedback is considered to solve all kinds of comprehensive problem firstly. Therefore, the consensus protocol with state feedback form is adopted:

$$u_i(t) = B_i^{\dagger}(A_s - A_i)x_i(t) + K_i \sum_{j=1}^{N} a_{ij}[x_j(t) - x_i(t)], t \geq 0, \ i \in \mathcal{V}, \tag{4}$$

where $B_i^{\dagger}$ is Moore-Penrose inverse of B_i, A_s and K_i, $i \in \mathcal{V}$ will be designed as follows.

Definition 4: For descriptor multi-agent system (2) with the fixed topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, protocol (4) is said to solve the admissible consensus problem (or system (2) achieves admissible consensus via protocol (4)) if the following conditions hold:

(i) The resultant closed-loop system via protocol (4) is impulse free;
(ii) $\lim\limits_{t \to \infty} \|x_j(t) - x_i(t)\| = 0, \ \forall i, j \in \mathcal{V}$.

Remark 3: Comparing with the consensus definition for normal multi-agent system (i.e. $E = I_n$ in system (2)), the admissible consensus definition adds the condition that the resultant closed-loop system via protocol (4) is impulse free. The main reasons for this are given as follows: On the one hand, the normal linear systems have always a unique solution, but for descriptor systems, the problem becomes complicated. Descriptor systems maybe have no solution, or have more than one solution. Investigation of the condition for existence and uniqueness of the solution to descriptor systems results in the concept of regularity. Since it can guarantee the existence and uniqueness of a solution to descriptor systems, regularity is a very important property for descriptor systems. On the other hand, different from that of normal linear systems, the response of descriptor systems may contain impulse terms. These impulse terms which may cause saturation of control and may even destroy the system are usually not expected to exist in most practical applications. Therefore, eliminating the impulsive behavior of descriptor systems via certain feedback control is an important fundamental problem in descriptor systems theory (Duan, 2010). According Definition 2, it can be obtained that a system is impulse free under the precondition which this system is regular. Based on the above reasons, Definition 4 adds the condition that the resultant closed-loop system via protocol (4) is impulse free.

The aim of this paper is to solve the following problem.

Problem 1: For descriptor multi-agent system (2) with the directed topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, design protocol (4) to solve the admissible consensus problem.

3. Analysis of admissible consensus for heterogeneous descriptor multi-agent system

Theorem 1: *For descriptor multi-agent system (2) with the directed topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, protocol (4) solves the admissible consensus problem of system (2) if and only if there exist $s \in \mathcal{V}$ and $K_i \in \mathbb{R}^{m\times n}$, $i \in \mathcal{V}$ satisfy that the following condition hold.*

(i) *The matrix pair $[I_N \otimes E,\ I_N \otimes A_s - \tilde{B}(\mathcal{L}_\mathcal{G} \otimes I_n)]$ is impulse free, where $\tilde{B} = \mathrm{diag}\{B_1K_1,\ B_2K_2\ \cdots,\ B_NK_N\}$.*

(ii) *The matrix pair $[I_{N-1} \otimes E,\ I_{N-1} \otimes A_s - \bar{B}(\mathcal{L}_{22} \otimes I_n) + (\mathbf{1}_{N-1}\mathcal{L}_{12}) \otimes (B_1K_1)]$ is stable, where $\mathbf{1}_{N-1} = [1\ 1\ \cdots 1]^T \in \mathbb{R}^{N-1}$, $\bar{B} = \mathrm{diag}\{B_2K_2,\ B_3K_3\ \cdots,\ B_NK_N\}$, $\mathcal{L}_{12} \in \mathbb{R}^{N-1}$ and $\mathcal{L}_{22} \in \mathbb{R}^{(N-1)\times(N-1)}$ are defined by $\mathcal{L}_\mathcal{G} = \begin{bmatrix} l_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$.*

Proof. Denote

$$\begin{aligned} x(k) &= [\,x_1^T(k)\ x_2^T(k)\ \cdots\ x_N(k)^T\,]^T, \\ \delta_i(k) &= x_i(k) - x_1(k),\ i \in \mathcal{V}\setminus\{1\}, \\ \delta(k) &= [\,\delta_2^T(k)\ \delta_3^T(k)\ \cdots\ \delta_N^T(k)\,]^T. \end{aligned}$$

According to $\mathrm{rank}B_i = n$, one has $B_i^\dagger B_i = I_n$. The closed-loop system which is made of system (2) and protocol (4) is as follows:

$$E\dot{x}_i(t) = A_s x_i(t) + B_iK_i \sum_{j=1}^{N} a_{ij}[x_j(t) - x_i(t)],\ i \in \mathcal{V}. \tag{5}$$

Using the definition of the element l_{ij} of Laplacian matrix $\mathcal{L}_\mathcal{G}$, system (5) can be written as

$$E\dot{x}_i(t) = A_s x_i(t) - B_iK_i \sum_{j=1}^{N} l_{ij}\delta_j(t),\ i \in \mathcal{V},$$

then it follows that the compact form of the closed-loop system is obtained:

$$(I_N \otimes E)\dot{x}(t) = [I_N \otimes A_s - \tilde{B}(\mathcal{L}_\mathcal{G} \otimes I_n)]x(t). \tag{6}$$

So it is easy to see that system (6) is impulse free if and only if the matrix pair $[I_N \otimes E,\ I_N \otimes A_s - \tilde{B}(\mathcal{L}_\mathcal{G} \otimes I_n)]$ is impulse free.

The state difference system of agent i and agent 1 is as follows:

$$E\dot{\delta}_i(t) = A_s\delta_i(t) - B_iK_i\sum_{j=2}^{N} l_{ij}\delta_j(t) + B_1K_1\sum_{j=2}^{N} l_{1j}\delta_j(t). \tag{7}$$

Thus, the compact form of system (7) can be described as

$$(I_{N-1}\otimes E)\dot{\delta}(t) = [I_{N-1}\otimes A_s - \bar{B}(\mathcal{L}_{22}\otimes I_n) + (\mathbf{1}_{N-1}\mathcal{L}_{12})\otimes (B_1K_1)]\delta(t). \tag{8}$$

It can be concluded that system (2) via protocol (4) achieves $\lim\limits_{t\to\infty}\|x_j(t)-x_i(t)\| = 0,\ \forall i,j\in\mathcal{V}$ if and only if $\lim\limits_{t\to\infty}\|\delta(t)\| = 0$ holds, that it is system (8) achieve asymptotically stable, which implies that the matrix pair $[I_{N-1}\otimes E,\ I_{N-1}\otimes A_s - \bar{B}(\mathcal{L}_{22}\otimes I_n)+(\mathbf{1}_{N-1}\mathcal{L}_{12})\otimes(B_1K_1)]$ is stable. Therefore, protocol (4) solves the admissible consensus problem if and only if the condition (i) and (ii) hold, simultaneously. The proof is completed. □

Corollary 1: *For descriptor multi-agent system (2) with the directed topology $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{A})$, protocol (4) solves the admissible consensus problem of system (2) if the topology $\mathcal{G}$ contains a directed spanning tree, and there exists $s\in\mathcal{V}$ such that $(E,\ A_s)$ is impulse free, and $(E,\ A_s,\ B_s)$ is stabilizable.*

Proof. Using the preconditions which $(E,\ A_s)$ is impulse free and $(E,\ A_s,\ B_s)$ is stabilizable , it can be concluded from Lemma 4 that descriptor Ricatti equation

$$A_s^TX + X^TA_s - X^TB_sB_s^TX + I_n = 0, \tag{9a}$$

$$E^TX = X^TE \geq 0 \tag{9b}$$

has the unique admissible solution P in the sense of E^TX, then it follows that $(E, A_s - B_sB_s^TP)$ is admissible.

Since the topology $\mathcal{G}$ contains a directed spanning tree, it is obtained that $\mathcal{L}_{\mathcal{G}}$ has only one zero eigenvalue. Let $\lambda_1 = 0,\ \lambda_i,\ i\in\mathcal{V}\setminus\{1\}$ be all eigenvalues of $L_{\mathcal{G}}$, $\theta = \min\limits_{2\leq i\leq N}\{\mathrm{Re}(\lambda_i)\}$ and $K_s = \max\{\frac{1}{2},\ \frac{1}{\theta}\}B_s^TP$. It can be concluded from Lemma 5 that $(E,\ A_s - \lambda_iB_sK_s)$ is admissible for $\forall i\in\mathcal{V}\backslash\{1\}$. Choose $K_i = B_i^{\dagger}B_sK_s$. Using $\mathrm{rank}B_i = n$, one obtains from Lemma 1 that $B_iK_i = B_sK_s$ holds for $\forall i\in\mathcal{V}$. Then it follows that

$$\begin{aligned}\tilde{B} &= \mathrm{diag}\{B_1K_1,\ B_2K_2\cdots,\ B_NK_N\} = I_N\otimes(B_sK_s),\\ \bar{B} &= \mathrm{diag}\{B_2K_2,\ B_3K_3\cdots,\ B_NK_N\} = I_{N-1}\otimes(B_sK_s).\end{aligned}$$

Thus,

$$\begin{aligned}&I_N\otimes A_s - \tilde{B}(\mathcal{L}_{\mathcal{G}}\otimes I_n) = I_N\otimes A_s - \mathcal{L}_{\mathcal{G}}\otimes(B_sK_s),\\ &I_{N-1}\otimes A_s - \bar{B}(\mathcal{L}_{22}\otimes I_n) + (\mathbf{1}_{N-1}\mathcal{L}_{12})\otimes(B_1K_1)\\ =&I_{N-1}\otimes A_s - (\mathcal{L}_{22} - \mathbf{1}_{N-1}\mathcal{L}_{12})\otimes(B_sK_s),\end{aligned}$$

where $\mathbf{1}_{N-1} = [1\ 1\ \cdots 1]^T\in\mathbb{R}^{N-1}$, $\mathcal{L}_{12}$ and $\mathcal{L}_{22}$ are defined as in Theorem 1.

Denote $H = \begin{pmatrix} 1 & 0 \\ \mathbf{1}_{N-1} & I_{N-1} \end{pmatrix}$. Using the definition of Laplacian $L_{\mathcal{G}}$ gives

$$H^{-1} L_{\mathcal{G}} H = \begin{pmatrix} 0 & \mathcal{L}_{12} \\ 0 & \mathcal{L}_{22} - \mathbf{1}_{N-1}\mathcal{L}_{12} \end{pmatrix}. \tag{10}$$

It can be obtained from the above equation that

$$\sigma(\mathcal{L}_{22} - \mathbf{1}_{N-1}\mathcal{L}_{12}) = \{\lambda_2,\ \lambda_3 \cdots,\ \lambda_N\}.$$

Hence there exists a nonsingular matrix F such that

$$F^{-1}(\mathcal{L}_{22} - \mathbf{1}_{N-1}\mathcal{L}_{12})F = J = \mathrm{diag}(J_1, \cdots, J_s),$$

where $J_k,\ k = 1, 2, \cdots, s$ are upper triangular Jordan blocks, whose principal elements consist of $\lambda_i,\ i \in \mathcal{V} \setminus \{1\}$, it follows that

$$\begin{aligned} &(F \otimes I_n)^{-1}(I_{N-1} \otimes E)(F \otimes I_n) = I_{N-1} \otimes E, \\ &(F \otimes I_n)^{-1}[I_{N-1} \otimes A_s - (\mathcal{L}_{22} - \mathbf{1}_{N-1}\mathcal{L}_{12}) \otimes (B_s K_s)](F \otimes I_n) \\ =&I_{N-1} \otimes A_s - J \otimes (B_s K_s). \end{aligned}$$

So one has

$$\begin{aligned} &\sigma[I_{N-1} \otimes E,\ I_{N-1} \otimes A_s - (\mathcal{L}_{22} - \mathbf{1}_{N-1}\mathcal{L}_{12}) \otimes (B_s K_s)] \\ =&\bigcup_{i=2}^{N} \sigma(E,\ A_s - \lambda_i B_s K_s). \end{aligned}$$

Then one obtains $[I_N \otimes E,\ I_N \otimes A_s - \tilde{B}(\mathcal{L}_{\mathcal{G}} \otimes I_n)]$ is impulse free if and only if $(E,\ A_s)$ and $(E,\ A_s - \lambda_i B_s K_s),\ i \in \mathcal{V} \setminus \{1\}$ are impulse free, and $[I_{N-1} \otimes E,\ I_{N-1} \otimes A_s - \bar{B}(\mathcal{L}_{22} \otimes I_n) + (\mathbf{1}_{N-1}\mathcal{L}_{12}) \otimes (B_1 K_1)]$ is stable if and only if $(E,\ A_s - \lambda_i B_s K_s),\ i \in \mathcal{V} \setminus \{1\}$ are stable. Therefore, it can be concluded from Theorem 1 that system (2) achieves admissible consensus via protocol (4) if and only if $(E,\ A_s)$ is impulse free and $(E,\ A_s - \lambda_i B_s K_s),\ i \in \mathcal{V} \setminus \{1\}$ are admissible. The proof is completed. $\square$

Based on Corollary 1, the following algorithm is provided to design protocol (4), which implies that Problem 1 will be solved under preconditions of Corollary 1.

Algorithm 1: *Input: the matrices* $E,\ A_i \in \mathbb{R}^{n\times n},\ B_i \in \mathbb{R}^{n\times m}$ *and* $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$*;*
Output: the gain matrices K *and* $K_i,\ i \in \mathcal{V}$.

(a) *Choose* $s \in \mathcal{V}$ *such that* $(E,\ A_s)$ *is impulse free and* $(E,\ A_s,\ B_s)$ *is stabilizable;*
(b) *Solve descriptor Racatti equation (9), and the admissible solution is denoted by* P*;*
(c) *Compute Laplacian matrix* $\mathcal{L}_{\mathcal{G}}$ *and the non-zero eigenvalues* $\lambda_i,\ i \in \mathcal{V} \setminus \{1\}$*. Denote* $\theta \triangleq \min\limits_{2\le i\le N}\{\mathrm{Re}(\lambda_i)\}$*. Set* $K_s = \max\{\frac{1}{2},\ \frac{1}{\theta}\}B_s^T P$*;*
(d) *Compute* $B_i^{\dagger}$*, and let* $K_i = B_i^{\dagger} B_s K_s,\ i \in \mathcal{V}$*;*
(e) *Output the gain matrices* A_s *and* $K_i,\ i \in \mathcal{V}$.

Remark 4: The details on solving Equation (9) is given in our previous work (Yang & Liu, 2014).

Remark 5: If θ obtained in step (c) is very small, the elements of the matrix K in step (d) will correspondingly become very large. In this case, the computed high-gain K is not usually acceptable in practical applications.

Remark 6: When $B_1 = B_2 \cdots = B_N = B$ and rank$B = n$ in system (2), protocol (4) is simplified as:

$$u_i(t) = B^{\dagger}(A_s - A_i)x_i(t) + K\sum_{j=1}^{N} a_{ij}[x_j(t) - x_i(t)], t \geq 0,\ i \in \mathcal{V}, \tag{11}$$

where $B^{\dagger}$ is Moore-Penrose inverse of B, A_s and K will be designed as follows. Corresponding to Theorem 1 and Corollary 1, the following corollaries are given. The proof is omitted.

Corollary 2: *For descriptor multi-agent system (2) with the directed topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, protocol (11) solves the admissible consensus problem of system (2) if and only if there exist $s \in \mathcal{V}$ and $K \in \mathbb{R}^{n\times n}$ satisfy that the following condition hold.*

(i) *The matrix pair $[I_N \otimes E,\ I_N \otimes A_s - (\mathcal{L}_{\mathcal{G}}) \otimes (BK)]$ is impulse free;*
(ii) *The matrix pair $[I_{N-1} \otimes E,\ I_{N-1} \otimes A_s - (\mathcal{L}_{22} + \mathbf{1}_{N-1}\mathcal{L}_{12}) \otimes (BK)]$ is stable, where $\mathbf{1}_{N-1} = [1\ 1\ \cdots 1]^T \in \mathbb{R}^{N-1}$, $\mathcal{L}_{12}$ and $\mathcal{L}_{22}$ are defined as in Theorem 1.*

Corollary 3: *For descriptor multi-agent system (2) with the directed topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, protocol (11) solves the admissible consensus problem of system (2) if the topology $\mathcal{G}$ contains a directed spanning tree, and there exists $s \in \mathcal{V}$ such that $(E,\ A_s)$ is impulse free, and $(E,\ A_s,\ B)$ is stabilizable.*

4. Numerical example

Example 1: Consider heterogeneous descriptor multi-agent system (2) consisting of $N = 3$ agents with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 5 & 1 \\ -2 & 1 & 3 \end{bmatrix},\ A_2 = \begin{bmatrix} 3 & 7 & 2 \\ -1 & 5 & -1 \\ -2 & 1 & 3 \end{bmatrix},\ A_3 = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & -1 \\ 2 & 3 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.1 & 2 & -1 & 1 \\ 3 & -1 & 0.6 & -3 \\ 2 & -1 & 0.5 & -1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 2 & -3 \\ 0.5 & 3 & 1 & -1 \end{bmatrix},\ B_3 = \begin{bmatrix} -0.1 & 2 & -1 & 1 \\ 5 & -1 & 0.5 & -1.5 \\ 1 & 3 & 1 & -1 \end{bmatrix},$$

and the topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, 3\}$, $\mathcal{E} = \{(2, 1),\ (2, 3),\ (3, 2)\}$ and $\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$.
Figs. 3 shows its communication topology. Obviously, the topology $\mathcal{G}$ has a spanning tree.
According to the steps in Algorithm 1, the following can be obtained:

(a) $A_s = \begin{bmatrix} 2.7 & 4.3 & 3.4 \\ 4.3 & 10.3 & 5.7 \\ 3.4 & 5.7 & 5.1 \end{bmatrix}$ and $s = 2$;

(b) $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

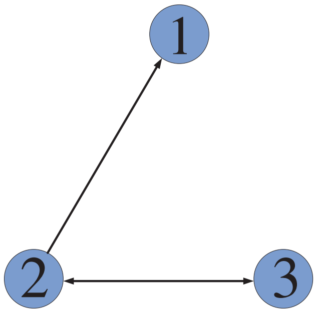

Figure 3. The communication topology

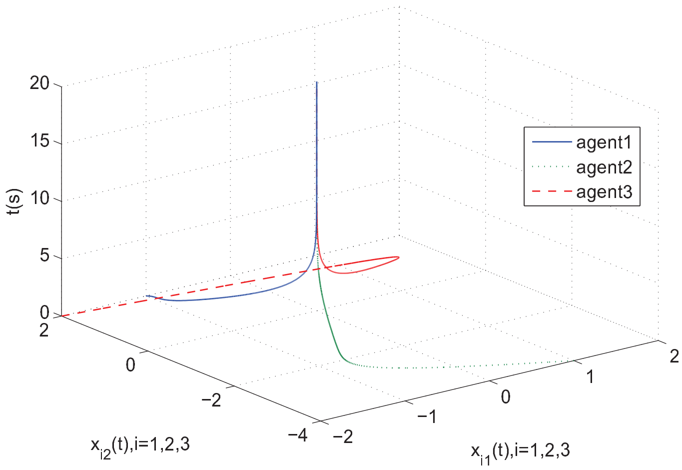

Figure 4. Trajectories of state x_{i1} and x_{i2}, $i=1,2,3$.

(c) $\mathcal{L}_{\mathcal{G}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix}$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\theta = 1$ and $K_s = K_2 = \begin{bmatrix} 1 & 1 & 0.5 \\ 3 & 4 & 3 \\ 1 & 3 & 1 \\ -2 & -5 & -1 \end{bmatrix}$;

(d) $B_1^{\dagger} = \begin{bmatrix} 0.3 & -0.1 & 0.7 \\ 0.4 & 0.3 & -0.5 \\ 0 & 0 & 0 \\ 0.1 & -0.6 & 0.8 \end{bmatrix}$, $B_3^{\dagger} = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0.2 & 0 & 0.2 \\ -0.6 & -0.1 & 0.4 \\ 0 & 0 & 0 \end{bmatrix}$,

$K_1 = \begin{bmatrix} 8.3 & 11 & 8 \\ -0.9 & -1.7 & -1 \\ 0 & 0 & 0 \\ 4.9 & 3.3 & 5.6 \end{bmatrix}$, $K_3 = \begin{bmatrix} 2.6 & 5.2 & 2.1 \\ 2.8 & 3.4 & 2.8 \\ 1.4 & 5.2 & 0.8 \\ 0 & 0 & 0 \end{bmatrix}$.

Choose the initial states as follows:

$$x_1(0) = \begin{bmatrix} -1 \\ 2 \\ 3.9 \end{bmatrix}, \; x_2(0) = \begin{bmatrix} 1 \\ 2 \\ 9.6 \end{bmatrix}, \; x_3(0) = \begin{bmatrix} -2 \\ -4 \\ -19 \end{bmatrix}.$$

The simulation results are presented in Figs. 4 to 7, respectively. Figs. 4 and 5 show state trajectories of heterogeneous descriptor multi-agent system (2) which indicates that system (2) achieves admissible consensus via protocol (4). Figs. 6 and 7 present trajectories of state differences of system (2) all tend to zero, which implies that system (2) achieves admissible consensus again by using Definition 4.

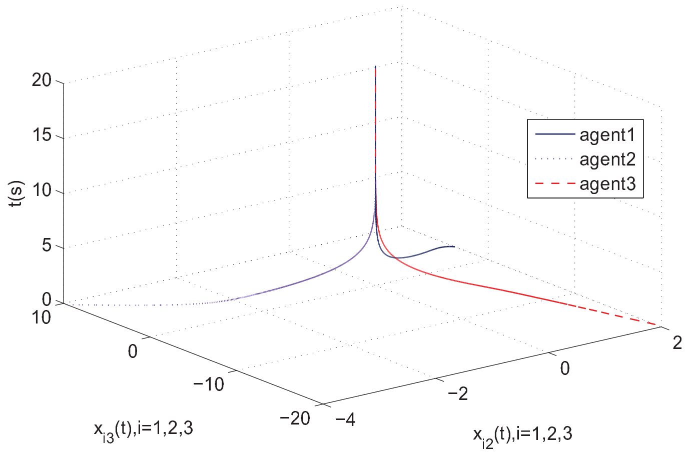

Figure 5. Trajectories of state x_{i1} and $x_{i3},\ i = 1, 2, 3$.

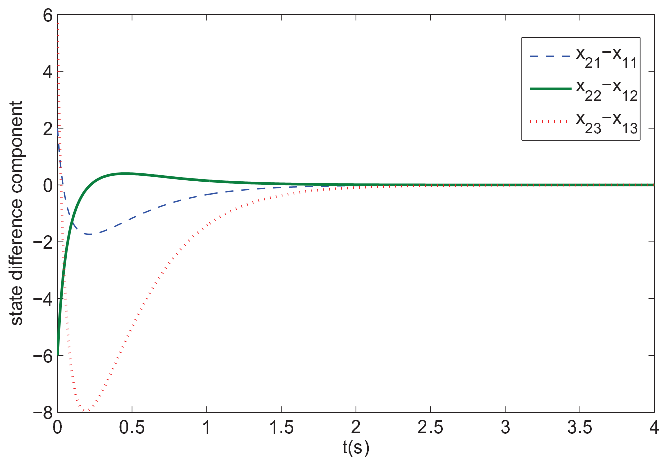

Figure 6. Trajectories of state difference $x_2 - x_1$

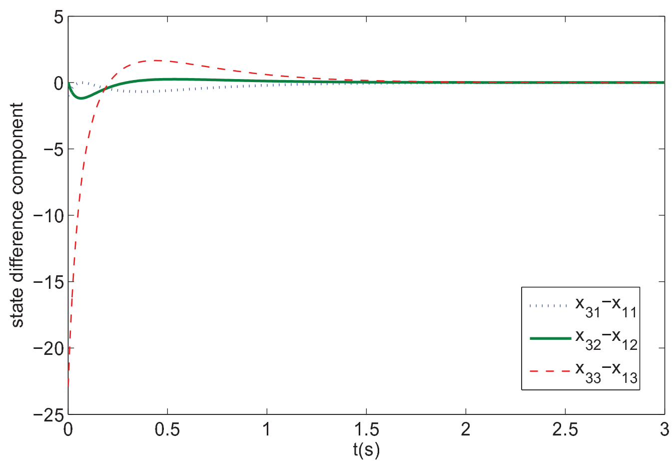

Figure 7. Trajectories of state difference $x_3 - x_1$

5. Conclusions

For heterogeneous descriptor multi-agent systems with directed topologies, this paper has solved the admissible consensus problem via the novel protocols. The consensus algorithm has been provided to design the distributed protocol. The given simulation results have successfully demonstrated the effectiveness of the given design approach. However, it is worth noticing that the study of admissible consensus for heterogeneous descriptor multi-agent systems without communication delays is a basic problem, which only serves as a stepping stone to investigate heterogeneous descriptor multi-agent systems with networked communication delays or more complicated topologies. The future research will study descriptor multi-agent systems with time-varying networked delays, stochastic or switching topologies, and agents described by switching systems (Zhang et al., 2011), hybrid systems (Xia et al., 2009; Xia, 2008) or Markovian jump systems (Li et al., 2014, 2015; Zhang et al., 2014), and so on.

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References

Ben-Israel, A. and Greville, T.N.E. (1974). *Generalized Inverses: Theorey and Applications.* New York: John Wiley.

Boughari, H. and Radhy, N.E. (2007). Regulation of linear continuous-time singular systems with constrained states and controls. *Int. J. Syst. Sci.*, 38, 689–698.

Cong, S. (2014). New stability criteria of linear singular systems with time-varying delay. *Int. J. Syst. Sci.*, 45, 1927–1935.

Dong, X.Z. (2014). Admissibility analysis of linear singular systems via a delta operator method. *Int. J. Syst. Sci.*, 45, 2366–2375.

Duan, G.R. (2010). *Analysis and Design of Descriptor Linear Systems.* New York: Springer.

Feng, Z. and Hu, G.P. (2014). Distributed tracking control for multi-agent systems under two types of attacks. In *Proceedings of the 19th IFAC World Congress*, pp. 24-29.

Guo, G., Ding, L. and Han, Q.L. (2014). A distributed event-triggered transmission strategy for sampled-data consensus of multi-agent systems. *Automatica*, 50, 1489–1496.

Hill, D.J. and Maareels, I.M.Y. (1990). Stability theory for differential/algebraic systems with application to power systems. *IEEE Trans. Circuits Syst.*, 37, 1416–1423.

Horn, R.A. and Johnson, C.R. (1990). *Matrix analysis.* Cambridge University Press.

Hsieh, C.S. (2014). H-infinity Kalman estimation for rectangular descriptor systems with unknown inputs. *IEEE Trans. Autom. Control*, 59, 826–832.

Huang, C.P. (2014). Stability analysis and controller synthesis for discrete uncertain singular fuzzy systems with distinct difference matrices in the rules. *Int. J. Syst. Sci.*, 45, 1830–1843.

Luenberger, D.G. and Arbel, A. (1977). Singular dynamic Leontief systems. *Econometrica*, 45, 991–995.

Li, F.B., Shi, P., Wu, L.G. and Michael, V. B. (2014). Quantized control design for cognitive radio networks modeled as nonlinear semi-Markovian jump systems. *IEEE Trans. Ind. Electron.*, DOI: 10.1109/TIE.2014.2351379.

Li, F.B., Shi, P., Wu, L.G. and Zhang, X. (2014). Fuzzy-model-based D-stability and nonfragile control for discrete-time descriptor systems with multiple delays. *IEEE Trans.Fuzzy Syst.*, 22, 1019–1025.

Li, F.B., Wu, L.G., Shi, P. and Lim, C.C. (2015). State estimation and sliding mode control for semi-Markovian jump systems with mismatched uncertainties. *Automatica*, 51, 385–393.

Li, F.B. and Zhang, X. (2012). A delay-dependent bounded real lemma for singular LPV systems with time-variant delay. *Int. J. Robust Nonlinear Control*, 22, 559–574.

Lin, P. and Jia, Y. (2010). Robust H_∞ consensus analysis of a class of second-order multi-agent systems with uncertainty. *IET Control Theory Appl.*, 4, 487–498.

Nguyen, M.H. and Tran, D.Q. (2014). Combination trust model for multi-agent systems. *International Journal of Innovative Computing, Information and Control*, 9, 2405–2420.

Pan, H., Nian, X.H. and Guo, L. (2014). Second-order consensus in multi-agent systems based on second-order neighbours' information. *Int. J. Syst. Sci.*, 45, 902–914.

Ren, W. and Beard, R.W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Trans. Autom. Control*, 50, 655–661.

Sedziwy, A. (2014). Effective graph representation supporting multi-agent distributed computing. *International Journal of Innovative Computing, Information and Control*, 10, 101–113.

Sastry, S.S. and Desoer, C.A. (1981). Jump behavior of circuits and systems. *IEEE Trans. Circuits Syst.*, 28, 1109–1124.

Su, Z., Zhang, Q.L. and Ai, J. (2013). Practical and finite-time fuzzy adaptive control for nonlinear descriptor systems with uncertainties of unknown bound. *Int. J. Syst. Sci.*, 44, 2223–2233.

Wang, H.J., Xue, A.K. and Lu, R.Q. (2014). New stability criteria for singular systems with time-varying delay and nonlinear perturbations. *Int. J. Syst. Sci.*, 45, 2576–2589.

Wu, Z. and Zhou, W. (2007). Delay-dependent robust H_∞ control for uncertain singular time-delay systems. *IET Control Theory Appl.*, 1, 1234–1241.

Xia, Y.Q., Boukas, E.K., Shi, P. and Zhang, J.H. (2009). Stability and stabilization of continuous-time singular hybrid systems. *Automatica*, 45, 1504–1509.

Xi, J.X., Shi, Z.Y. and Zhong, Y.S. (2012). Admissible consensus and consensualization of high-order linear time-invariant singular swarm systems. *Physica A*, 391, 5839–5849.

Xi, J.X., Yu, Y., Liu, G.B. and Zhong, Y.S. (2014). Guaranteed-cost consensus for singular multi-agent systems with switching topologies. *IEEE Trans. Circuits Syst. I: Regular Papers*, 61, 1531–1542.

Xia, Y.Q., Zhang, J.H. and Boukas, E.K. (2008). Control for discrete singular hybrid systems. *Automatica*, 44, 2635–2641.

Yang, X.R. and Liu, G.P. (2012). Necessary and sufficient consensus conditions of descriptor multi-agent systems. *IEEE Trans. Circuits Syst. I: Regular Papers*, 59, 2669–2677.

Yang, X.R. and Liu, G.P. (2014). Necessary and sufficient consensus conditions of descriptor multi-agent systems. *IET Control Theory Appl.*, 4, 487–498.

Yang, X.R. and Liu, G.P. (2014). Consensus of singular multi-agent systems based on networked predictive control. In *Proceedings of the 19th IFAC World Congress*, pp. 8329-8334.

Yang, D.M., Zhang, Q.L. and Yao, B. (2004). *Singular Systems*. Science Press: Beijing.

Yang, X.R., Zhang, R.Z. and Zhang, X. (2010). Regulation of rectangular descriptor systems with constrained states and controls. In *Proceedings of 2010 Chinese Conf. Decision Control*, pp. 1005–1010.

Yu, W.W., Zhou, L.X., Yu, H., Lu, J.H. and Lu, R.Q. (2013). Consensus in multi-agent systems with second-order dynamics and sampled data. *IEEE Trans. Autom. Control*, 9, 2137–2146.

Zhang, L.X., Cui, N.G., Liu, M. and Zhao, Y. (2011). Asynchronous filtering of discrete-time switched linear systems with average dwell time. *IEEE Trans. Circuits Syst. I: Regular Papers*, 58, 1109–1118.

Zhang, Y.Q., Shi, P., Nguang, S.K. and Song, Y.D. (2014). Robust fnite-time H_∞ control for uncertain discrete-time singular systems with Markovian jumps. *IET Control Theory Appl.*, 8, 1105–1111.

Zhang, Y.M., Zhang, Q.L., Tanaka, T. and Yan, X.G. (2014). Positivity of continuous-time descriptor systems with time delays. *IEEE Trans. Autom. Control*, 59, 3093–3097.

Zhu, J.D. Lu, J.H. and Yu , X.H. (2013). Flocking of multi-agent non-holonomic systems with proximity graphs. *IEEE Trans. Circuits Syst. I: Regular Papers*, 60, 199–210.

Zhang, Y. and Tian, Y.P. (2014). Allowable delay bound for consensus of linear multi-agent systems with communication delay. *Int. J. Syst. Sci.*, 45, 2172–2181.

Zheng, Y.S., Zhu, Y.R. and Wang, L. (2014). Finite-time consensus of multiple second-order dynamic agents without velocity measurements. *Int. J. Syst. Sci.*, 45, 597–588.